Mechanism Design for Online Resource Allocation: A Unified Approach

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Joint Work with

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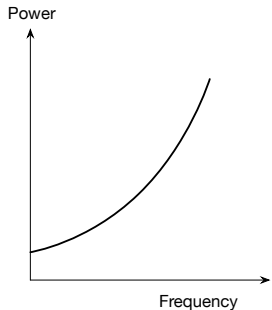$6 $3 $4

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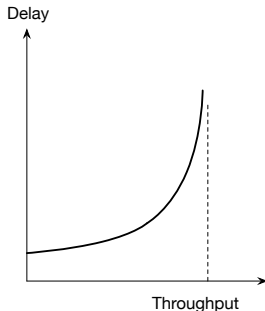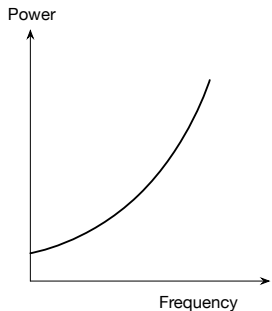
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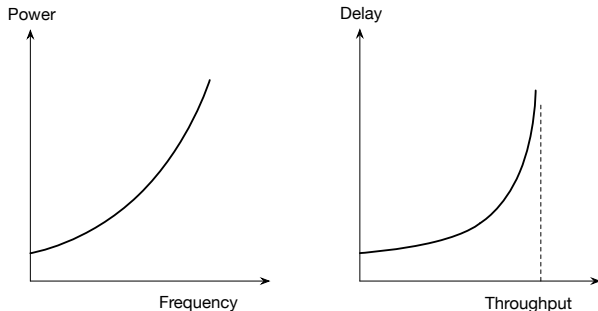
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Strictly-convex f

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Competitive Ratio

- **Offline Setting**: knows all arrival information:

$$S_{\text{offline}}(\mathcal{A}) = \sum_{n \in \{1,2,\cdots\}} v_n x_n^* - f\left(\sum_{n \in \{1,2,\cdots\}} r_n x_n^*\right),$$

where $\mathcal{A} = \{(v_1, r_1), (v_2, r_2), \cdots\}$ denotes an arrival instance.

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where $\mathcal{A} = \{(v_1, r_1), (v_2, r_2), \cdots\}$ denotes an arrival instance.

- **Online Setting**: develop posted prices, $\{p_n\}_{\forall n}$, whose competitive ratio:

$$\alpha \triangleq \max_{\text{all possible } \mathcal{A}} \frac{S_{\text{offline}}(\mathcal{A})}{S_{\text{online}}(\mathcal{A})}$$

is bounded by a constant independent of the number of agents.

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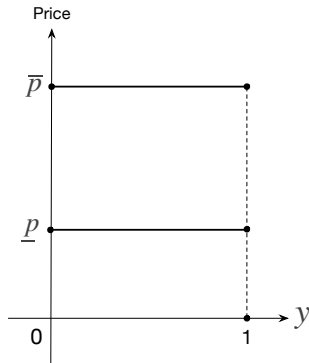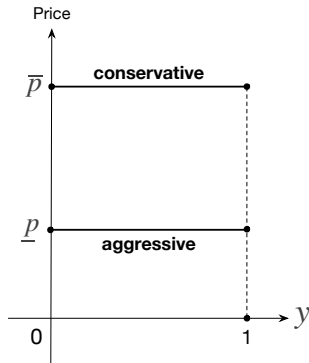
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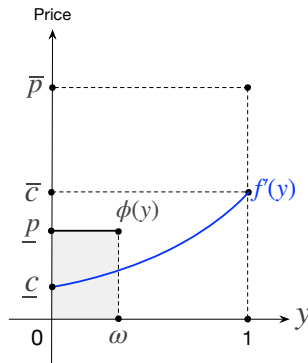
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competitive ratio $\alpha \longrightarrow 1$
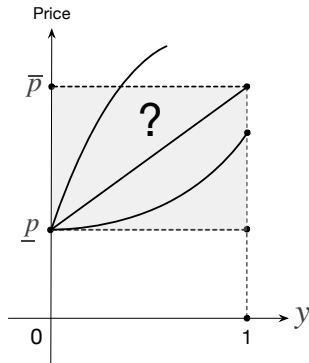
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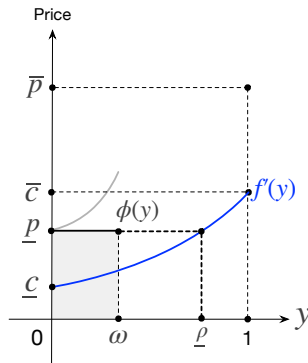- bounded value density: $\underline{p} \leq \frac{v_n}{r_n} \leq \bar{p}$

Design Principles

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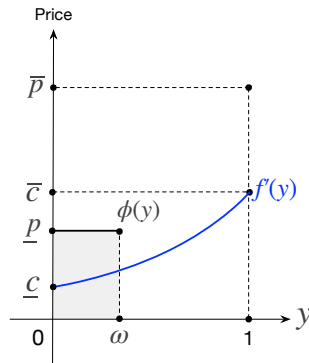
$$\boxed{\alpha \geq \frac{S_{\text{offline}}(\mathcal{A}_{\underline{p}})}{S_{\text{online}}(\mathcal{A}_{\underline{p}})} = \frac{\underline{p}\underline{\rho} - f(\underline{\rho})}{\underline{p}r_1 - f(r_1)} \xrightarrow{r_1 \to 0} +\infty}$$

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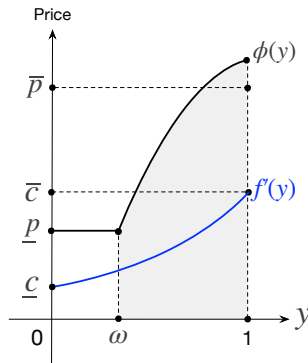
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- **P3**: sell the resource at a profitable price.
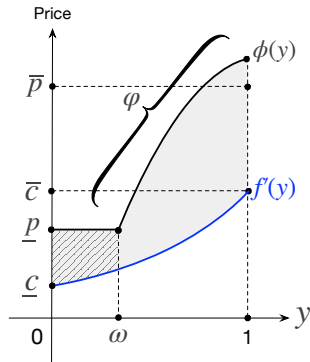 - $\phi(y) > f'(y)$ must hold for all $y \in [0, 1]$.
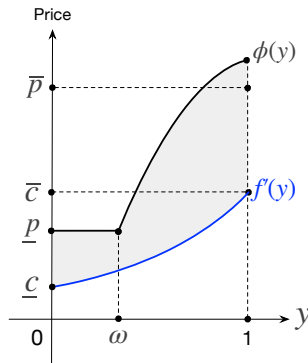
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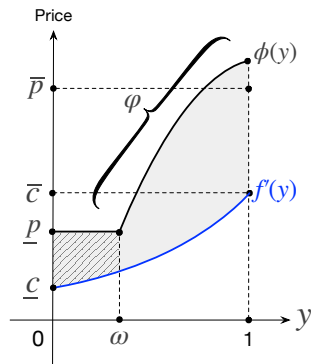
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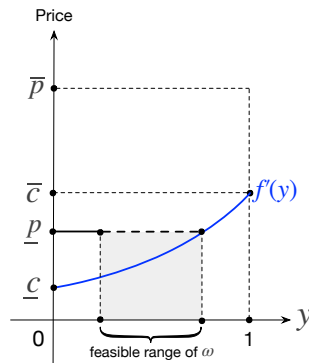

The **flat segment** ω and the **increasing segment** $\varphi(y)$.

Main Results: Sufficient Conditions

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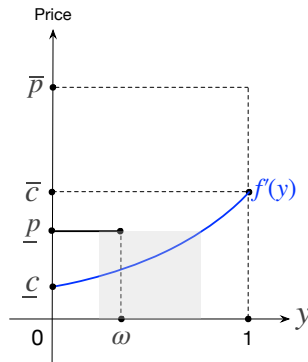where $h(p)$ is defined as $h(p) \triangleq \max\limits_{y \in [0,1]} py - f(y)$.

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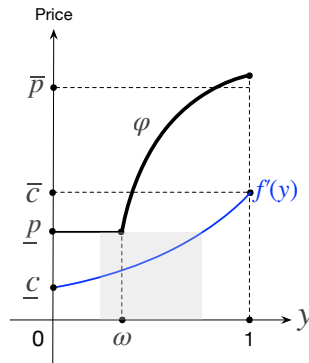
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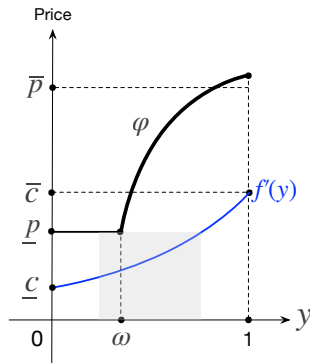
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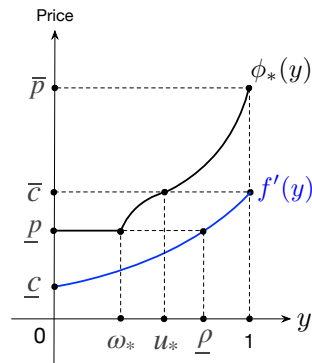

Online Primal-Dual Approach

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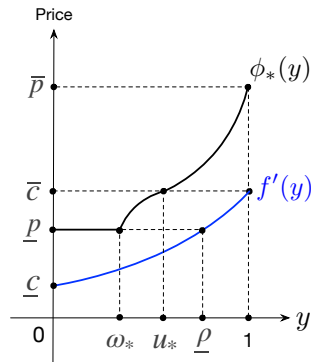$$\alpha_* = \frac{h(\underline{p})}{\underline{p}\omega_* - f(\omega_*)}$$

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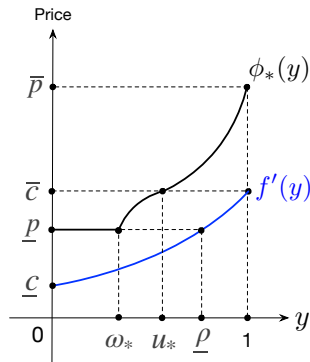
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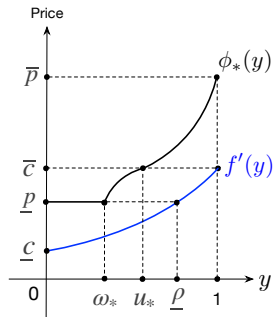$$\alpha_* = \frac{h(\underline{p})}{\underline{p}\omega_* - f(\omega_*)}$$

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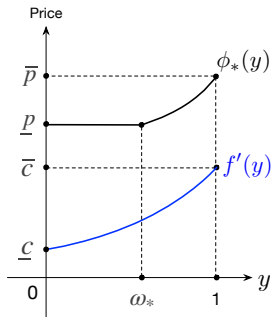

Remarks: i) ω_* depends on $f, \underline{p}$, and $\bar{p}$ and ii) $\varphi_*'(y) = \alpha_* \cdot \Phi(\varphi_*, y)$.
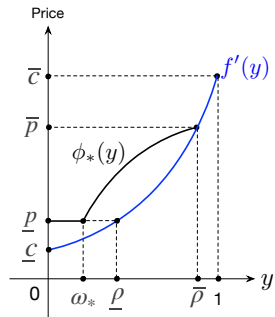
General Cases: Strictly-Convex Supply Costs

Case-1: $\underline{c} < \underline{p} < \bar{c} < \bar{p}$

Case-2: $\underline{c} < \bar{c} \leq \underline{p} \leq \bar{p}$

Case-3: $\underline{c} < \underline{p} \leq \bar{p} \leq \bar{c}$

Special Cases: Zero and Linear Supply Costs

- Given $\mathcal{S} = \{f, \underline{p}, \bar{p}\}$ with $f(y) = qy$, where $q \geq 0$, there exists a unique ϕ_*:

$$\phi_*(y) = \begin{cases} \underline{p} & \text{if } y \in [0, \omega_*), \\ (\underline{p} - q) \cdot \exp\left(\frac{y}{\omega_*} - 1\right) + q & \text{if } y \in [\omega_*, 1], \\ +\infty & \text{if } y \in (1, +\infty), \end{cases}$$

such that PM_{ϕ_*} is α_*-competitive, where α_* and ω_* are given by

$$\alpha_* = 1 + \ln\left(\frac{\bar{p} - q}{\underline{p} - q}\right), \quad \omega_* = \frac{1}{\alpha_*}.$$

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Remark: the logarithmic competitive ratio is not new, see [42] for details.

A Unified Approach

- zero supply costs: $\alpha_* = 1 + \ln\left(\frac{\bar{p}}{\underline{p}}\right)$ when $f(y) = 0$ (e.g., [41], [42]).

- linear supply costs: $\alpha_* = 1 + \ln\left(\frac{\bar{p}-q}{\underline{p}-q}\right)$ when $f(y) = qy$ with $q > 0$ (e.g., [41]).

- **strictly-convex supply costs:** $\alpha_* = \dfrac{h(\underline{p})}{\underline{p}\omega_* - f(\omega_*)}$.
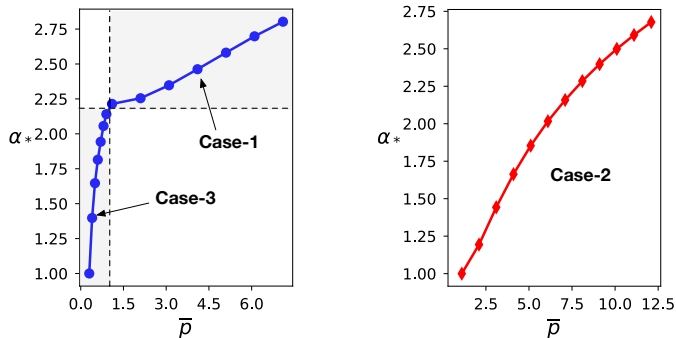
A Case Study: Quadratic Supply Costs

Figure: Illustration of α_* when $f(y) = \frac{1}{2}y^2$. Left: $\underline{p} = 0.3$. Right: $\underline{p} = 1.1$.

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- A **general** model that can be extended to more complex settings.
    - multi-knapsack problems.
    - multi-unit auctions and combinatorial auctions.
    - $\cdots$

Thank You

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Homepage: https://xiaoqitan.org